

No-cloning of Orthogonal States in Composite Systems

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The no-cloning principle tells us that non-orthogonal quantum states cannot be cloned, but it does not tell us that orthogonal states can always be cloned. We suggest a situation where the cloning transformations are restricted, leading to a novel type of no-cloning principle. In the case of a composite system made of two subsystems: if the subsystems are only available one after the other then there are various cases when *orthogonal states* cannot be cloned. Surprising examples are given, which give a radically better insight regarding the basic concepts of quantum cryptography.

The no-cloning theorem describes one of the most fundamental non-classical properties of quantum systems. It states that an unknown quantum state cannot be cloned [1,2]. Assuming that such a quantum pure state ρ can be cloned leads to a violation of unitarity of quantum mechanics. Even if there is some information about the state (e.g., it is either ρ_0 or ρ_1) it cannot be cloned in the general case. This means that we cannot create a copying device which gets the unknown state (ρ_p) as an input and produces two copies of it at the output. Another version of the no-cloning theorem [3] states that any attempt of learning something regarding the input state of the copying device (even an attempt of making a very faint imprint) will necessarily induce some disturbance in the output state; This principle presents a very interesting variation of the so called “uncertainty principle” since it applies to an individual system; see [4] for more details.

Let $\rho_0 \equiv |\phi_0\rangle\langle\phi_0|$ and $\rho_1 \equiv |\phi_1\rangle\langle\phi_1|$ be two non identical pure states provided by the producer (Alice), and suppose that these states are known to the person (Eve) who attempts to clone them. Let Eve receive one of them (ρ_p) — without knowing which one — as an input of her cloning (copying) device, and assume the initial normalized pure state of the cloning device is E . Then, a success to clone, that is, to create a product state $\rho_p\rho_p$ from the unknown input state ρ_p , is described by

$$E\rho_p \longrightarrow E_p' \rho_p \rho_p \quad (1)$$

where the dimensionality of the primed system E' is smaller than the dimensionality of E . This process violates unitarity if the states are non-orthogonal: unitarity promises that $\text{Tr}(EE)\text{Tr}(\rho_0\rho_1) = \text{Tr}(E_0'E_1')\text{Tr}(\rho_0\rho_1)\text{Tr}(\rho_0\rho_1)$; using the normalization conditions we get $\text{Tr}(EE) = 1$ and $\text{Tr}(E_0'E_1') \leq 1$, so for non-identical states with $\text{Tr}(\rho_0\rho_1) < 1$, cloning is impossible unless $\text{Tr}(\rho_0\rho_1) = 0$. Hence, non-identical states can be cloned only if they are orthogonal. In the cloning process above, Eve knows the set of states ρ_p , and therefore, in case they are identical she can clone them; the

cloning device creates the state $\rho = \rho_0 = \rho_1$ without measuring anything. In the more effective version of the no cloning theorem [3] the states changes according to

$$E\rho_p \longrightarrow E_p\rho_p . \quad (2)$$

In this case unitarity promises us that $\text{Tr}(EE)\text{Tr}(\rho_0\rho_1) = \text{Tr}(\rho_0\rho_1)\text{Tr}(E_0E_1)$, leading to $\text{Tr}(E_0E_1) = 1$ for (non-identical) non-orthogonal states. Thus E_0 and E_1 are identical and can provide no information on ρ_p . We shall refer to this process as *no-imprint principle* to distinguish it from the *no-cloning principle* of Eq. (1). If E_0 and E_1 are not identical (hence, provide information) in an imprint process, then the output states cannot be identical to the input states ρ_p . See [4] for detailed analysis of information vs. disturbance in case of non-zero disturbance in an imprint process.

These conclusions apply also when ρ_p are mixed states, telling us that only identical or orthogonal states ρ_p can be cloned using the first process [5] or imprinted using the second process. However, the situation is more delicate since such processes do not provide a complete description of Eve's possible strategies when ρ_p are mixed states! The no-cloning of quantum mixed states was recently analyzed, and it was shown that, while commuting non-orthogonal quantum mixed states cannot be cloned [using Eq. (1)], they can still be *broadcast* [5]: Eve can create a state χ_p such that

$$\chi_p \neq \rho_p \otimes \rho_p \quad (3)$$

but which satisfies

$$\text{Tr}_E[\chi_p] = \text{Tr}_A[\chi_p] = \rho_p . \quad (4)$$

A similar extension of Eq.(2) to achieve a “broadcast-imprint” process is straight forward. Surprisingly, it is not restricted to commuting density matrices [6], and a complete analysis is still missing. Due to Eq. (3) (or its

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counterpart when performing a broadcast-imprint process of mixed states), the state of the cloning device is entangled with the state of the system.

These no-cloning theorems prove that non-orthogonal states cannot be cloned, and that cloning of orthogonal states is possible if arbitrary unitary transformation can be chosen. In this work we suggest to *restrict* the allowed unitary transformations. We show that, as result of this restriction, there are orthogonal states which *cannot be cloned*. We suggest a particular restriction, where ρ_0 and ρ_1 are two orthogonal states of a composite system and the subsystems from which the system is composed are only available (to the cloning device) one after the other. Other restrictions are also possible, and lead to very fascinating examples [7].

The restriction of the type we use here is typical in quantum key distribution. We show that “no cloning of orthogonal states” is the basic principle used in many quantum key distribution schemes, rather than the standard no-cloning arguments, as was previously argued and believed. Hence, we shed new light on the possible basic concepts which are at the roots of secure quantum key distribution.

Let the system A in Alice’s hands be composed of two subsystems A_1 and A_2 , such that the possible states provided by Alice, $\Phi_p(A_1A_2)$, are orthogonal to each other, in the Hilbert space of the composite system. If the two subsystems were provided to Eve together she could clone the states. However, the subsystems are provided to Eve only one after the other so she cannot access the second one while she holds the first. Let $\rho_p(A_1) = \text{Tr}_{A_2}(\Phi_p)$ be the reduced density matrices of the first subsystem, and assume that the two (or more) possible states are non-identical and non-orthogonal. Consider Eve’s possible strategies when she holds the first subsystem A_1 . Clearly, if she changes $\rho_p(A_1)$ before letting it go her cloning attempt fails since she will have no access to that subsystem in the future. Thus, she cannot use the cloning process or imprint process. However, she can still use a broadcast-cloning process or a broadcast-imprint process. We only need to verify that these processes shall not allow her to clone in our scenario: Indeed, if the states of the first subsystem are non-orthogonal, Eve might be able to achieve a broadcast, but the state of the first subsystem necessarily becomes entangled with Eve’s state, hence cannot be fully entangled (or fully correlated) with the second subsystem anymore. Therefore, although the state $\rho_p(A_1)$ does not change in the broadcast process, the state Φ_p necessarily changes, and noise is induced. The last thing we should worry about is that Eve will not be able to clone by obtaining the entire information from the second subsystem, meaning that $\rho_p(A_2) = \text{Tr}_{A_1}(\Phi_p)$ must be non orthogonal as well.

Thus, based on the previous discussion, we reach a novel *no cloning principle for orthogonal states*. — The two (or more) orthogonal states $\rho_p(A_1A_2)$ of the system

composed of (A_1) and (A_2) cannot be cloned if the reduced density matrices of the subsystem which is available first (say A_1)

$$\rho_p(A_1) = \text{Tr}_{A_2}[\rho_p(A_1A_2)] \quad (5)$$

are non orthogonal and non identical, and if the reduced density matrices of the second subsystem are non orthogonal.

The first case we study is the case where the two subsystems are entangled, such that the first subsystem A_1 is in one of two commuting mixed states. Let A_1 and A_2 be two qubits with $|0_b\rangle$ and $|1_b\rangle$ the basis vectors of the b ’th qubit. Let the initial states be the two orthogonal states [8]

$$\psi_0 = \cos \alpha |0_1 \otimes 1_2\rangle + \sin \alpha |1_1 \otimes 0_2\rangle \quad (6)$$

$$\psi_1 = \sin \alpha |0_1 \otimes 1_2\rangle - \cos \alpha |1_1 \otimes 0_2\rangle, \quad (7)$$

and α is known to the cloner. In case $\alpha = 0$ or $\alpha = \pi/2$ all data is already contained in the first particle. In terms of cloning of the first subsystem – the two possible reduced states are orthogonal, hence can be cloned. In case of $\alpha = \pi/4$ (so that $\cos \alpha = \sin \alpha = 1/\sqrt{2}$) the two possible reduced states of the first subsystem are identical. Thus, Eve can release a dummy qubit (denoted by E_3) of her own, entangled with another one (denoted by E_4), say in a state $\psi_0 = (1/\sqrt{2})|0_3 \otimes 1_4\rangle + (1/\sqrt{2})|1_3 \otimes 0_4\rangle$, while keeping subsystem A_1 . The reduced state of the dummy particle is equal to the state of the first subsystem. Later on, after receiving subsystem A_2 and learning the state of the combined system A_1A_2 , Eve can change the state of (E_3E_4) by a simple transformation on particle E_4 alone. In terms of cloning of the first system – the two possible states are the same, the completely mixed state, hence can be cloned. For all other α , there is no strategy for Eve to learn the data, or even to get some information unless disturbing the state. The reduced density matrix of the first particle (particle 1) are

$$\rho_0 = \cos^2 \alpha |0\rangle\langle 0| + \sin^2 \alpha |1\rangle\langle 1| \quad (8)$$

and

$$\rho_1 = \sin^2 \alpha |0\rangle\langle 0| + \cos^2 \alpha |1\rangle\langle 1|. \quad (9)$$

The two density matrix commutes, but they are not orthogonal nor identical. Therefore they can be broadcast, but they can not be cloned, meaning that the resultant state of A_1 and E can be $\chi_p(A_1C)$ such that $\text{Tr}_E[\chi_p(A_1E)] = \rho_p$ and $\text{Tr}_{A_1}[\chi_p(A_1E)] = \rho_p$ but $\chi_p(A_1C)$ is not a tensor product of these two matrices. Therefore, Eve’s system is entangled with A_1 , and some noise is necessarily introduced since the resulting (three-particle) state is not a tensor product of Alice’s (two-particle) state and Eve’s state.

This example (in a somewhat different form which uses Fock states) was suggested recently by Koashi and Imoto [8] for quantum key distribution. It was suggested as a generalization and modification of a previous scheme of Goldenberg and Vaidman [9]. Goldenberg and Vaidman (GV) were the first to realize that one can use orthogonal states for quantum key distribution. Their work [9] emphasizes that non-orthogonal states are not crucial in quantum cryptography, but it does not discuss it in terms of no-cloning of the reduced density matrices of the first subsystem as we do here. GV's scheme is using random timing, since it uses the case of $\alpha = \pi/4$ which, without random timing, is insecure. Hence, it actually uses (as is explained in [8]) three orthogonal states, where the third one is

$$\psi_2 = |0_1\rangle|0_2\rangle, \quad (10)$$

(which is the vacuum state in their description) with a reduced density matrix

$$\rho_2 = |0\rangle\langle 0|. \quad (11)$$

This state ψ_2 is used only for error verification. [To observe that indeed [9] used three states, one must describe their work using Fock states $|0^{(a)}\rangle$ etc. in arms a and b]. Note that in this case, the three possible reduced density matrices of the first particle are still commuting, but they are *not identical*, hence can be broadcast but not cloned.

In a comment to [9], Peres [10] emphasizes that, as far as Alice is concerned, previous schemes for quantum cryptography such as the original (four-state) key distribution scheme of [11] also use orthogonal states, since, in the four-state scheme Alice chooses one of two orthogonal states in a basis of her choice. Peres also emphasizes the need of a second stage, where classical information is provided to Eve (regarding the basis in the four-state scheme or the random timing in GV scheme) when Eve has no longer access to the quantum data. See also Goldenberg-Vaidman reply [12]. Note that we do not consider the case of spatial separation, but only the case of “time separation”.

Using our new form of the no cloning argument we provide a stronger statement: in both the GV scheme and the four-state scheme the states which are *transmitted through Eve* are orthogonal! This is a very surprising result, since (a)—it tells us that entanglement is not vital for preventing cloning of orthogonal states, and since (b)—it was always believed that the four-state scheme uses four non-orthogonal states (see explanation in [12] for instance).

In the following we show striking similarities between GV scheme and the four-state scheme, which are far beyond the similarity shown by Peres. To achieve this understanding we present a non-standard, fully quantum description of the four-state scheme, which however, is completely equivalent to the standard description. We

claim that the standard belief that quantum key distribution is based upon the no-cloning theorem of Eq. (2) is inappropriate: In quantum key distribution Alice's states are sent to another person (Bob) who must be able to learn them (at least sometimes). The states are sent through Eve. It is not enough to prevent Eve from learning the encoded bits, but we also need to make sure that Bob can learn them. Thus, a standard no-cloning argument does not suffice as the basis of quantum key distribution. Indeed, using a complete quantum description, we show that our no-cloning argument for orthogonal states is actually used in the standard four-state scheme of [11].

Let 0 and 1 form a basis of two dimensional Hilbert space and let $0^x = (1/\sqrt{2})[0 + 1]$ and $1^x = (1/\sqrt{2})[0 - 1]$. Let Alice prepare one of the following four states $|\phi_{0_z}\rangle = |0_1 0_2\rangle$, $|\phi_{0_x}\rangle = |0_1^x 1_2\rangle$, $|\phi_{1_z}\rangle = |1_1 0_2\rangle$, and $|\phi_{1_x}\rangle = |1_1^x 1_2\rangle$, which are all orthogonal to each other. The first qubit (qubit 1) is in one of four pure states $|0\rangle$; $|1\rangle$; $(1/\sqrt{2})|0+1\rangle$; $(1/\sqrt{2})|0-1\rangle$, which are the states used in the conventional form of [11], and the second qubit carries classical information telling whether the first qubit was prepared in one basis or the other.

Consider the two mixed states

$$\chi_0 = |\phi_{0_z}\rangle\langle\phi_{0_z}| + |\phi_{0_x}\rangle\langle\phi_{0_x}| = |0_1 0_2\rangle\langle 0_1 0_2| + |0_1^x 1_2\rangle\langle 0_1^x 1_2| \quad (12)$$

and

$$\chi_1 = |\phi_{1_z}\rangle\langle\phi_{1_z}| + |\phi_{1_x}\rangle\langle\phi_{1_x}| = |1_1 0_2\rangle\langle 1_1 0_2| + |1_1^x 1_2\rangle\langle 1_1^x 1_2|, \quad (13)$$

which are used to encode 0 and 1 respectively, and are sent to Bob through Eve. [Note that combining each pair of states into one mixture is a result of the fact that the secret bit (the data that should be cloned) in [11] is defined by these mixed states]. The two mixed states χ_p are orthogonal, since each of the pure states which decompose one of the matrices is orthogonal to the states which decompose the other. The reduced density matrices of particle 1 (when particle 2 is traced out from the states χ_p) again have the form

$$\rho_0 = \cos^2 \alpha |0'\rangle\langle 0'| + \sin^2 \alpha |1'\rangle\langle 1'| \quad (14)$$

and

$$\rho_1 = \sin^2 \alpha |0'\rangle\langle 0'| + \cos^2 \alpha |1'\rangle\langle 1'| \quad (15)$$

(in a basis, denoted by the prime, which is exactly between the standard (z) basis and the x basis used above, and is also known as the Breidbart basis), with $\cos^2 \alpha = (1 + 1/\sqrt{2})/2$. These density matrices are commuting but not orthogonal. Therefore the two density matrices can be broadcast but not cloned, and any attempt of cloning them will create noise.

Note that we ignored the irrelevant step in which Bob is telling Alice his choice of basis. This step is required in the original protocol due to technological limitations, and can be eliminated once Bob keeps the first particle (in a quantum state) till receiving the second. However, it is crucial that Bob receives the first particle *before* Alice sends the second and communication from Bob to Alice is required to verify this.

This far we have seen the impossibility to clone three orthogonal states when two are entangled [9], the impossibility to clone two orthogonal entangled states [8] and the impossibility to clone four non-entangled (product) states [10] or two non-entangled (sum of product) density matrices (our description of [11]). Let us search for simpler cases, with non entangled orthogonal states. Such a case will make the best use of the new no-cloning theorem stated in this work. Furthermore, such orthogonal states which cannot be cloned are useful for quantum key distribution, and might be simpler to implement, or to analyze, relative to the other schemes.

It is impossible to use only two product pure states: Let $|\phi_0\rangle = |0_1'0_2''\rangle$ (which is general since the basis is arbitrary), and $|\phi_1\rangle$ a state which is orthogonal to it; Clearly, $|\phi_1\rangle = |1_1'\chi_2\rangle$ or $|\phi_1\rangle = |\chi_11_2''\rangle$ (with arbitrary state χ , and with 1 orthogonal to 0 in each primed or double-primed basis) are the only two possibilities to choose a two-particle state orthogonal to $|\phi_0\rangle$. For any of these choices the entire information can be cloned by Eve by cloning the appropriate particle (e.g. the first particle in the first choice of $|\phi_1\rangle$); similar argument is true when using higher dimensions or larger number of particles.

Note that we did not describe the scheme of [13] in terms of cloning, since a more complicated involvement of the third party (Bob) is needed for its description. Two-way communication between Alice and Bob plays a vital role in this scheme, and it is not clear yet if our new no-cloning principle suffices to explain why this scheme works.

The natural simplest (achievable) possibilities are to use only three orthogonal pure states or two pure orthogonal states and one mixed state which is orthogonal to both; as far as quantum key distribution is concerned, let two pure non entangled states carry the secret key, and the third non entangled state (pure or mixed) provide the protection from cloning (this is called “data rejected protocol”): The states

$$|\phi_0\rangle = |0_10_2\rangle ; \quad |\phi_1\rangle = |1_10_2\rangle \quad (16)$$

will be two states which carry the key (in the first qubit), and in addition the state

$$|\phi_2\rangle = |0_1^{(x)}1_2\rangle , \quad (17)$$

or alternatively, an equal mixture of this state and $|1_1^{(x)}1_2\rangle$,

$$\chi_2 = |0_1^{(x)}1_2\rangle\langle 0_1^{(x)}1_2| + |1_1^{(x)}1_2\rangle\langle 1_1^{(x)}1_2| , \quad (18)$$

provide the protection from cloning. Such schemes (especially the second) are as simple as [11] as far as practical implementation is concerned, but might be much simpler to analyze, since Eve can only obtain information in one basis.

In this work we discussed the impossibility to clone orthogonal states of combined systems in various case, based on the impossibility to clone non orthogonal mixed states. We presented a unified formalism for several schemes in quantum key distribution based on our new no-cloning argument for orthogonal states. We corrected an unjustified claim regarding the role of Eq. (2) in quantum key distribution, and solved a dispute regarding the use of orthogonal states in quantum key distribution. Finally we suggested the simplest schemes which are based on this principle.

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